DESIGN OF ROBUST ENERGY CONTROL FOR CART - INVERTED PENDULUM

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ABSTRACT

The objective is to design a complete control system for the Swing-up and Stabilizing Control of cart - inverted pendulum. In particular, this work outlines the effectiveness of a particular Swing-up method based on feedback linearization and Total Energy Shaping (TES) considerations. The basic idea to solve the problem of driving the pendulum from down or any to upright position has been always to inject the necessary energy to do it. The power of modern state-space techniques for the analysis and control of MIMO systems is also investigated and state-feedback controller is employed for stabilizing the inverted pendulum. Thus, the stabilization approach consists in a first step that stabilizes the down subsystem via kinetic energy shaping, and a second step extends the solution to the upper subsystem via forwarding technique. And the robustness of this control method is confirmed by simulation under the influence of external rod disturbances.

KEYWORDS**:** *Total Energy Shaping (TES), Swing-Up and Stabilization, Cart- Pendulum*

INTRODUCTION

Being an under-actuated mechanical system and inherently open loop unstable with highly non-linear dynamics, the inverted pendulum system is a perfect test-bed for the design of a wide range of classical and contemporary control techniques [1]. Its applications range widely from robotics to space rocket guidance systems. Originally, these systems were used to illustrate ideas in linear control theory such as the control of linear unstable systems. Their inherent non-linear nature helped them to maintain their usefulness along the years and they are now used to illustrate several ideas emerging in the field of modern non-linear control [2].

A Single rod Inverted Pendulum (SIP) consists of a freely pivoted rod, mounted on a motor driven cart. With the rod exactly centered above the motionless cart, there are no sidelong resultant forces on the rod and it remains balanced as shown in Fig.1a. In principle it can stay this way indefinitely, but in practice it never does [3]. Any disturbance that shifts the rod away from equilibrium, gives rise to forces that push the rod farther from this equilibrium point, implying that the upright equilibrium point is inherently unstable as shown in Fig. 1b. Under no external forces, the rod would always come to rest in the downward equilibrium point, hanging down as shown in Fig. 1c.This is called the pendant position.

This equilibrium point is stable as opposed to the upright equilibrium point [4-6]. The control task is to swing up the pendulum from its natural pendant position and to stabilize it in the inverted position, once it reaches the upright equilibrium point[7]. The cart must also be homed to a reference position on the rail. All this is achieved only by moving the cart back and forth within the limited cart travel along the rail. The inverted pendulum system belongs to the class of under-actuated mechanical systems having fewer control inputs than degrees of freedom [8-10]. This renders the control task more challenging making the inverted pendulum system a classical benchmark for testing different control techniques [11-12].

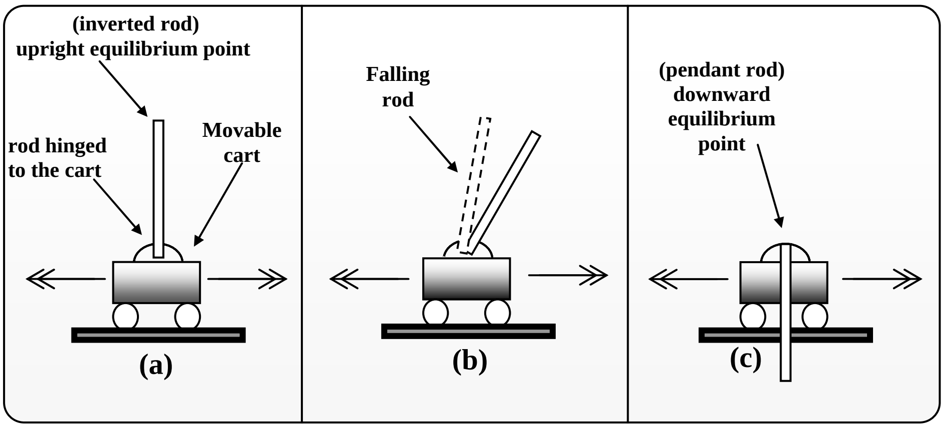

Fig.1. Equilibrium points

This section includes the introduction and importance of the project. Section 2 deals with the mathematical dynamic model of the system used both for the computer simulations (Matlab) and for the mathematical design of the controllers. Section 3 describes the main steps in the design of the control algorithms. Section 4 presents the simulation results and discussions and finally conclusions are drawn in Section 5.

MATHEMATICAL MODELING

The dynamic model of the whole system consists of two separate sub-models, namely: the non-linear model of the inverted pendulum, and a linear model of the Permanent Magnet (PM) DC motor powering the cart. This division was adopted in order to keep the non-linear dynamic equations of the inverted pendulum as simple as possible, which is imperative for the design of the non-linear swing-up controller.

The non-linear inverted pendulum model considers the force on the cart as the input, and the angle of the pendulum and cart displacement as the outputs. The motor model considers the motor terminal voltage as its input and the shaft torque as its output. Both models are derived separately and the resulting dynamic equations are then used in the design stage to develop two different control systems operating simultaneously at well distinct bandwidths.

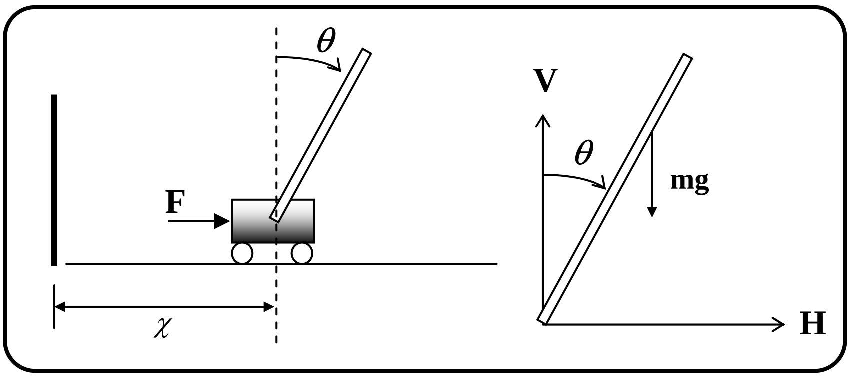

Fig. 2 Rod and cart setup and its force diagram

Non-Linear Dynamic Model

Referring to Fig.2 and applying Newton's 2nd law at the centre of gravity of the pendulum along the horizontal &vertical components yields,

$$V - mg = m \frac{d^2}{dt^2}(L\cos\theta) \quad (1)$$

$$\mathrm{H} = \mathrm{m}\frac{\mathrm{d}^2}{\mathrm{d}t^2}(x + L\sin\theta) \quad (2)$$

Taking moments about the centre of gravity yields the torque equation,

$$I\ddot{\theta} + c\dot{\theta} = VL\sin\theta - HL\cos\theta \quad (3)$$

Applying Newton's 2nd law for the cart yields,

$$F - H = M\ddot{x} + k\dot{x} \quad (4)$$

Where m is the mass at the Centre Of Gravity (COG) of the pendulum; M is the mass of the cart; L is the distance from the COG of the pendulum to the pivot; x is the horizontal displacement of the cart; *g* is the gravitational acceleration; θ is the rod angular displacement; k is the cart viscous friction coefficient; c is the pendulum viscous friction coefficient; I is the moment of inertia of the pendulum about the COG; V & H are the vertical & horizontal reaction forces on the rod and F is the horizontal control force on the cart. Combining (1) to (4), the non-linear mathematical model of the cart and pendulum system is obtained and is given by (5) and (6).

$$\ddot{\theta} = \frac{1}{I + L^2 m}\left[Lm\left(g\sin\theta - \ddot{x}\cos\theta\right) - c\dot{\theta}\right] \quad (5)$$

$$\ddot{x} = \frac{1}{M+m}\left[F - Lm\left(\ddot{\theta}\cos\theta - \dot{\theta}^2\sin\theta\right) - k\dot{x}\right] \quad (6)$$

State-Space Linearized Model:

Equations (5) and (6) were used to model the open-loop inverted pendulum (motor dynamics not included) during simulations. The same non-linear model was used for the design of the non-linear swing-up controller. However, for the design of the linear state-feedback controller, used for stabilization, a linearized version of these equations was used. The inverted position of the pendulum corresponds to the unstable equilibrium poin$\left(\theta,\dot{\theta}\right)$t = (0, 0). This corresponds to the origin of the state space.

In the neighborhood of this equilibrium point, both θ and $\dot{\theta}$ are very small (in rad & rad/sec respectively).

For small angles of θ and $\dot{\theta}: \sin(\theta) = \theta, \cos(\theta) = 1$ and $\dot{\theta}^2\ \theta = 0$.Using these approximations in (5) and (6), the mathematical model linearized around the unstable equilibrium point of the inverted pendulum is obtained, and given by (7) and (8).

$$\ddot{\theta} = \frac{1}{I + L^2 m}\left[Lm\left(g\theta - \ddot{x}\right) - c\dot{\theta}\right] \quad (7)$$

$$\ddot{x} = \frac{1}{M+m}\left[F - Lm\ddot{\theta} - k\dot{x}\right] \quad (8)$$

To get these two equations into valid state space linear form both $\ddot{x}$ and $\ddot{\theta}$ must be functions of lower order terms only. Hence, $\ddot{x}$ must be substituted for in (7) using (8), and similarly $\ddot{\theta}$ substituted for in (8) using (7). Writing the resulting equations in matrix form, the linearized state-space model is obtained and is given by the matrix linear equations (9) and (10).

$$\dot{s} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & -kv_2 & \left(\dfrac{-(Lm)^2 g v_2}{I + L^2 m}\right) & \left(\dfrac{Lmcv_2}{I + L^2 m}\right) \\ 0 & 0 & 0 & 1 \\ 0 & \left(\dfrac{Lmkv_1}{M + m}\right) & \left(Lmgv_1\right) & -cv_1 \end{bmatrix} s + \begin{bmatrix} 0 \\ v_2 \\ 0 \\ \left(\dfrac{-Lmv_1}{M + m}\right) \end{bmatrix} F \tag{9}$$

$$y = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} s \tag{10}$$

where state vector,

$$s = \begin{bmatrix} x & \dot{x} & \theta & \dot{\theta} \end{bmatrix}^T \tag{11}$$

Output vector,

$$y = \begin{bmatrix} x & \theta \end{bmatrix}^T \tag{12}$$

$$v_1 = \frac{M + m}{I(M + m) + L^2 m M} \tag{13}$$

and $v_2 = \dfrac{I + L^2 m}{I(M + m) + L^2 m M}$ (14)

Permanent Magnet DC Motor Dynamics:

The mathematical model for the motor, considering the motor terminal voltage as input and the shaft torque as output, is given by the transfer function in (15).

$$\frac{T(s)}{V_T(s)} = \frac{\dfrac{K_t}{L} s}{\left(s^2 + \dfrac{R}{L} s + \dfrac{K_t^{\,2}}{LJ}\right)} \tag{15}$$

Where L is the terminal inductance of the motor; R is the terminal resistance; K_t is the torque constant; J is the rotor inertia; V_T is the terminal voltage and T is the developed torque.

ENERGY CONTROLLER DESIGN

A new approach to swing-up control is proposed based on an energy control method (TES). That is, noting that the energy of the pendulum can be controlled according to the sign condition of the cart acceleration, we develop a method for controlling the cart acceleration under a limited travel of the cart. The design procedure of the controller that controls the energy of the pendulum is quite simple, which mainly consists of constructing a servo system having a low -pass property and using a sinusoidal reference input generated from the pendulum trajectory. When the pendulum is close to the inverted vertical, a full-state feedback controller takes over the control and stabilizes the whole system.

Design Strategy

The inverted pendulum control was split in two main phases: the swing-up phase and the stabilizing phase. The former uses a non-linear controller to swing-up the pendulum, keeping the cart within a limited travel range on the rail. The latter uses a linear state-feedback controller to stabilize the pendulum in the inverted position once it approaches the upper unstable equilibrium point. It is also required to home the cart to a reference point on the rail, once the pendulum is stabilized. A transition algorithm switches smoothly from one control phase to the other.

Swing-Up Control:

Swing up control technique aims at swinging up the pendulum, while keeping the cart within a limited horizontal travel on the rail [1]. This is achieved by satisfying a particular mathematical condition, derived from the mechanical energy equations of the pendulum, while constructing a linear servo system, using a sinusoidal reference input generated from the pendulum trajectory. The total mechanical energy of the pendulum *V*, and its derivative $\dot{V}$, are given by (16) and (17).

$$V = \frac{1}{2} m L^2 \dot{\theta}^2 + m g L (1 - \cos\theta) \tag{16}$$

$$\dot{V} = \mathrm{m\,L}\,\dot{\theta}\left(\cos\theta\right)\ddot{x} \tag{17}$$

From (16) and (17), it is clear that *V* can be increased or decreased or shaped (TES) by changing the sign (*sgn*) of $\ddot{x}$ in accordance with that of $\dot{\theta}\cos\theta$. If sgn ($\ddot{x}$) = sgn ($\dot{\theta}\cos\theta$) then $\dot{V}>0$, similarly if sgn ($\ddot{x}$) = -sgn ($\dot{\theta}\cos\theta$) then $\dot{V}<0$. Hence, energy can be pumped into the pendulum by generating $\ddot{x}$ (acceleration on the cart) satisfying the sign conditions listed above. However, one cannot concentrate on swinging-up the pendulum only, without considering the finite cart travel (limited range for *x*). Therefore, $\ddot{x}$ has to be controlled while keeping the constraint on *x* in mind. Basically, the design method proposed in [1] suggests; constructing a control law such that the resulting closed loop system is linear (through feedback linearization) and of the form of a second order servo system for *x*, having a sinusoidal reference input to ensure the desired bounded nature of *x*. This reference input is derived from ($\theta,\dot{\theta}$), and generates $\ddot{x}$ satisfying the sign condition given above. This is done in order to control *V* to the prescribed value corresponding to the energy of the pendulum at the upright equilibrium point. Since the pair of ($\theta,\dot{\theta}$) that makes *V* equals to the desired value is not unique, the upright equilibrium point cannot be stabilized using only this control method. For this reason a different control law is utilized when the pendulum approaches the upright equilibrium point. This is referred to as stabilizing control.

Stabilizing Control:

This control method is based on state-space pole placement design techniques using the linearized model of the inverted pendulum. This implies that the stabilizing control by itself will only ensure local stability, in the vicinity of the upright equilibrium point, the point about which the equations were linearized. The pole-placement technique permits the design of a linear controller that achieves arbitrary desired closed loop poles. The desired poles should be chosen wisely such that some desired closed loop characteristics are achieved. The final control law, from this design, is the result of a matrix multiplication between the state vector *s* and a gain matrix of compatible dimensions K^T, such that $F = -K^T s$. In this particular design, a small settling time and a high damping ratio were required. To meet these specifications, the closed loop poles were placed at,
$s = \mu_i$ ($i = 1, 2, 3, 4$),
Where,

$$\mu_1 = -0.238 + j0.1624, \mu_2 = -0.238 - j0.1624,$$

$$\mu_3 = -2, and\, \mu_4 = -2$$

Basically, μ_1 and μ_2 are a pair of dominant closed-loop poles with damping factor $\xi = 0.826$ and natural frequency =0.2882 rad/sec, resulting in a settling time of approximately 8 seconds. The other two poles are located far to the left of the dominant pair of closed loop poles and therefore, their effect over the overall response is minimal.

Transition Algorithm

An intermediate algorithm was designed to switch from the swing-up controller to the stabilizing controller and vice versa, depending on the state variables θ and $\dot{\theta}$ .Actually, this algorithm performs a smooth transition from one control law to the other by averaging the outputs of the two controllers in the transition region. This avoids what is known as hard switching, this may upset the system due to parameter uncertainties and un-modeled dynamics. Basically, if θ and $\dot{\theta}$ .are both close to zero, only the stabilizing controller is used. Similarly if they are much higher than zero, only the swing-up algorithm is used. In between these two extremes a region was created, the transition region, in which both algorithms are processed and weighted accordingly, leading to soft switching.

Proposed System Structure:

The proposed control structure consists of an energy confinement based swing up strategy and a stabilizing state feed back control. When the pendulum moves close to the vertical position the control is transferred from energy based method to state feedback.

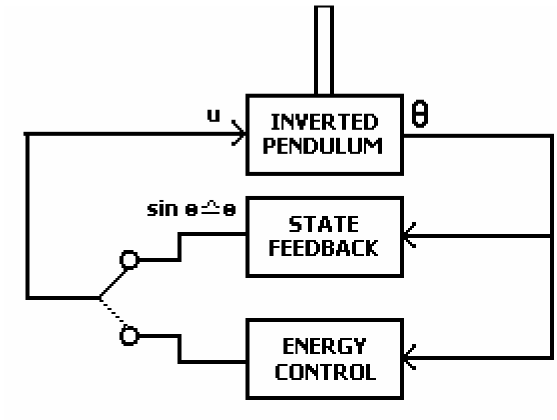

Fig. 3 Proposed control structure

RESULTS AND DISCUSSIONS

The Furuta [2] cart-inverted pendulum system has been simulated for the following specifications:

M = 1.900 kg; m = 0.500 kg; L = 0.45 m ; I = 0.00338 kg-m^2; J = 18e-7 kg-m^2; c = 0.01; k = 0.165; g = 9.81 N/m^2; K = 100; E_0 = 0; K_t = 0.024 Nm/A; R = 2.4 m; L = 300e-6 m; B = 1.1e-6; K_b = 0.02

The controllers discussed in section 3, were simulated using the nonlinear model of the Furuta Pendulum. The inner loop dynamics were neglected in these simulations. This is valid since in practice the inner loop was designed to have a much higher bandwidth than the pendulum dynamics.

System under Study

The non-linear model of the plant is designed based upon the equations of the system, derived from the basic equations of motion. The model was developed in SIMULINK toolbox of MATLAB.

Results of the cart-pendulum System

The response of linear and angular displacement of the inverted pendulum is shown in Fig. 4 and 5. The inverted pendulum system has two distinct equilibrium point concerning the pendulum angle. The first equilibrium point is at the upright position which is unstable and the
second equilibrium point is at the hanging position. From the figure, we know that the pendulum settles at the downward equilibrium position.

However, we focus on the problem of raising the pendulum from its stable equilibrium to the vicinity of its unstable equilibrium state. Clearly this is a problem of how to increase the total energy of the pendulum

(TES). Hence, the design of nonlinear controller of Furuta-inverted pendulum is basically based on energy shaping. This is accomplished by the feedback linearization during swing-up.

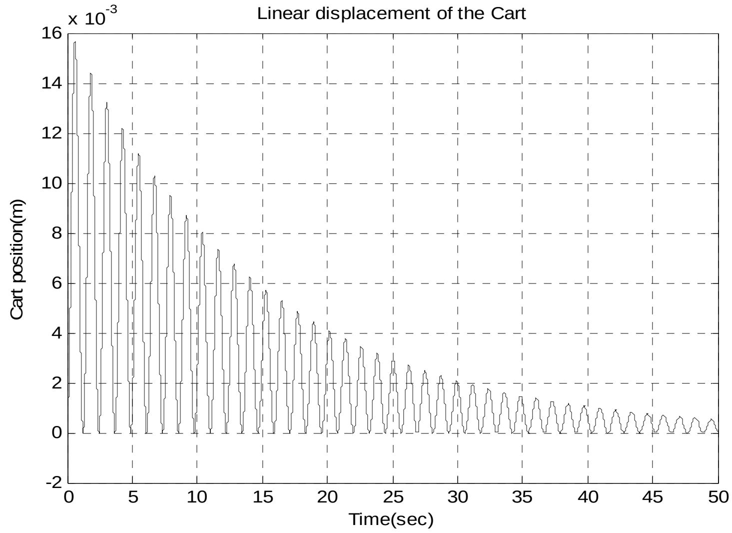

Fig. 4. Displacement of the cart in m Vs Time

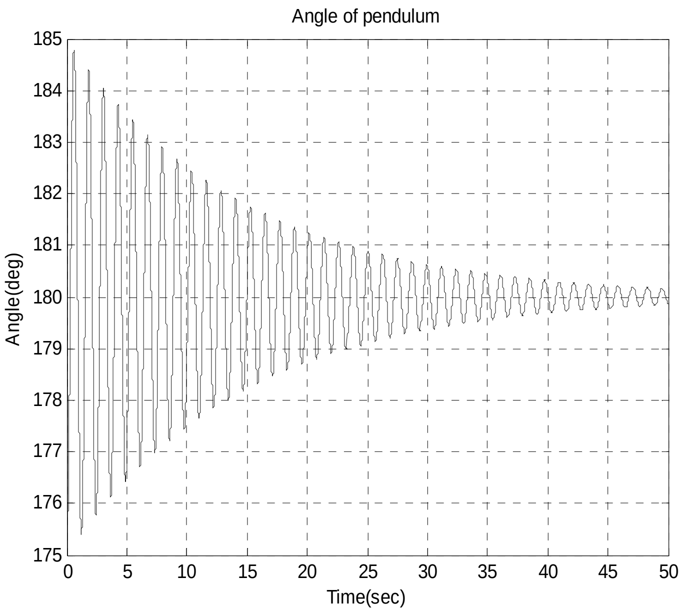

Fig. 5 Angular position of the pendulum in Degrees Vs Time

Swinging Up and Stabilizing Responses

The simulation results for the inverted pendulum with the controller are given here. From the simulation results, we can ensure that the controller performs properly so as to make the inverter pendulum state at its upright position. For the purpose of validating this controller through simulations, physical limitations and constraints have been added to the simulation model. These are, namely: (i) the cart displacement is such that $|r| \leq 1$m; (ii) the voltage to the motor should be $\leq \pm$ 20V due to the physical limitations of the analogue amplifier; and in addition, models of data sampling, zero order-hold used in the microcomputer and of the finite difference differentiation for speed computations are included in the simulation model.

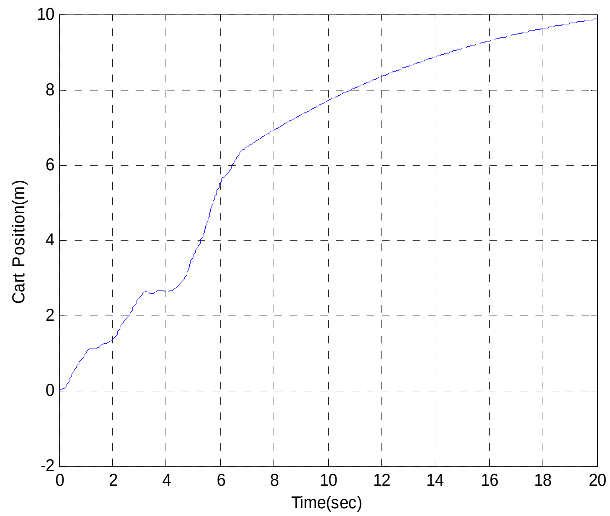

Fig. 6. Displacement of the cart in m Vs Time ($\theta_0 = 90°$)

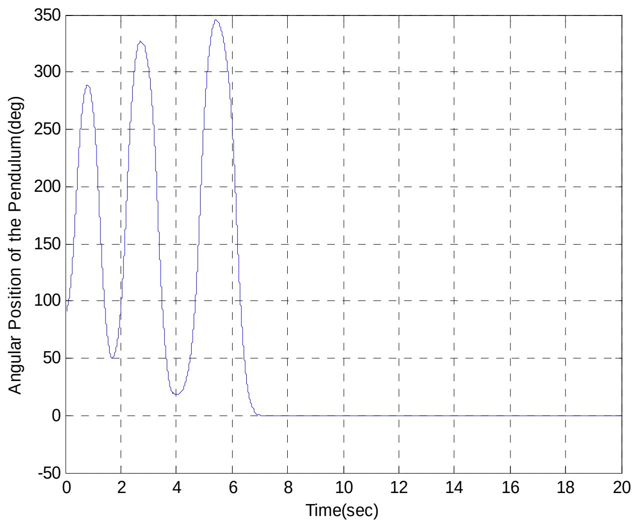

Fig.7. Angular position of the pendulum in Degrees Vs Time(when $\theta_0 = 90°$)

The study of the response of the unstable linear system to small perturbations (initial conditions) about its vertical position results in a system time constant of 68 ms. The recommended sampling time interval for the computer data acquisition system, to prevent aliasing, is at least three times less than the lowest time constant of the system [9]. Initially, the sampling time interval was chosen to be 15 ms. However, due to system uncertainties, the response of the nonlinear system is faster (around 451ns). Hence, after extended trials, the sampling time interval to give best stability was found to be 12ms. Since the very initial time period of the experiment is the most important one, the performance index above generates a time varying controller which allows the system states to be more robust to uncertainties when *t* is small. For the purpose of real time experiment, an average time constant optimal feedback law [10] could be used instead.

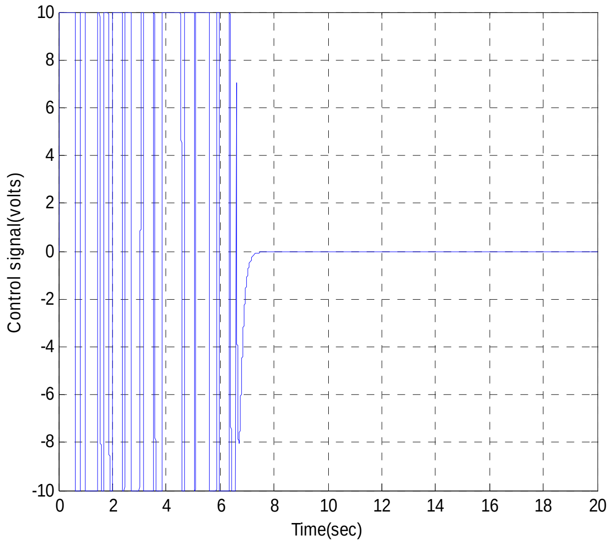

Fig. 8. Control Signal of Swinging up control in Voltage Vs Time (when $\theta_0 = 90^o$)

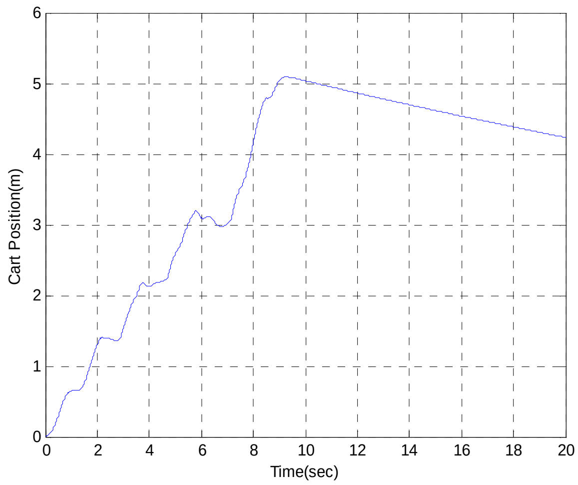

Fig. 9. Displacement of the cart in m Vs Time (when $\theta_0 = 180^o$)

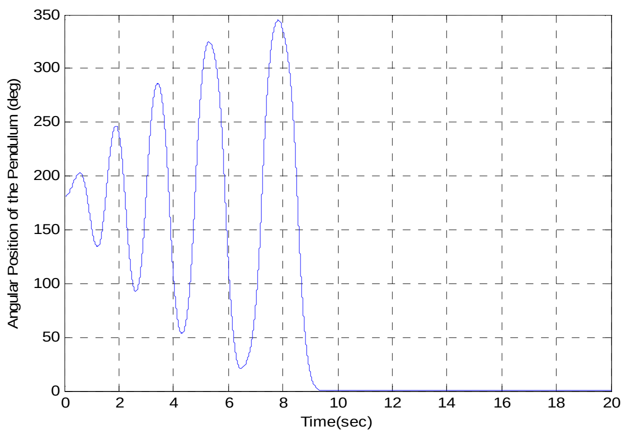

Fig. 10. Angular position of the pendulum in Degrees Vs Time ($\theta_0 = 180^o$)

Figures show the simulation plots for θ and x during swing-up and stabilization. Initially the pendulum is in the pendent position. It swings-up gradually, responding to the bounded oscillations of the cart. Up to 3.91 seconds the swing-up controller is in control. Then the transition algorithm takes over till 3.99 seconds. The state-feedback controller takes over completely for the rest of the time, stabilizing the pendulum in the inverted position and homing the cart to the reference point on the rail. Note that the pendulum swings-up and stabilizes in less than 8.7 seconds

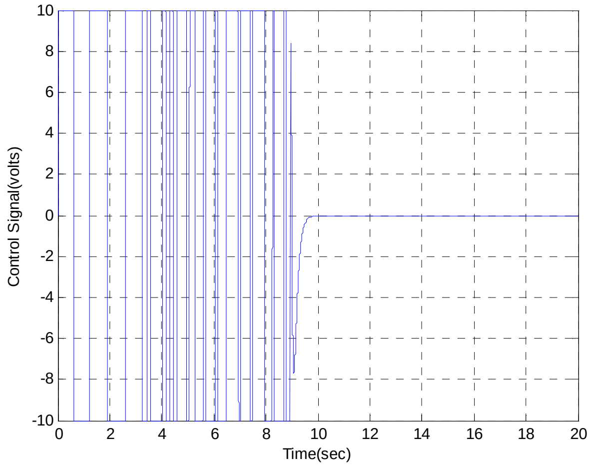

Fig. 11. Control Signal of Swing-up control in Voltage Vs Time (θ_0 = 180º)

The response shown in Fig. 8. is the control signal for the swing up and stabilizing control when the length of inverted pendulum is 45cm. It is seen from Fig. 8 that during the swing up state, the control signal is larger than the control signal in the stabilizing state. This is relevant because the large amount of energy is required to swing the pendulum up to its upright position and the small amount energy is only required to stabilize the pendulum.

From the above results, we can ensure that the pendulum can be swung up from the natural pendant position to upright position around by 9.7 seconds by energy controller. When the upright position has been reached, the system is switched to stabilizing control mode to stabilize the pendulum in its upright position. If the energy is sufficiently large, the pendulum can be brought to the upright position with one swing and two switches of the control signal.

The control signal uses its maximum value until the desired energy is obtained and is then set to zero. For lower accelerations the pendulum has to swing several times. The cart also homed back quickly to a reference position on the rail. The controlled inverted pendulum was proven to be highly robust for external rod disturbances and the controller exhibited stability for both *x* and θ given any initial conditions.

CONCLUSION

`The pendulum is subjected to the energy based control (Total Energy Shaping) to swing up to the vertical position. Once it is brought close to the unstable vertical position, the control is transferred from swing up to the linear state feed back control and the pendulum is positioned in the vertical position. The above concepts were simulated and verified through MATLAB simulations. The results presented in section 4 verify that the system designed was successful. The control task stated in section 1 was completely fulfilled, i.e. the pendulum swung-up from its natural pendant position and stabilized in the inverted open-loop unstable position. Total Energy Shaping (TES) based control is a effective way to swing-up a pendulum. The behavior of such systems depends critically on one parameter, the maximum acceleration of the pivot. The case of swinging up a simple cart-pendulum has been treated in detail. However, the method is also suitable for swinging up of two pendulums on a cart and swinging up a double pendulum and triple link pendulum. A controller design technique which includes all system constraints (linear and nonlinear) should also be developed. This technique should result in a controller which could be easily implemented and does not restrict the sampling time interval of the Simulation.

REFERENCES

1. Yoshida, K (1999) Swing-up control of an inverted pendulum by energy based methods, proceedings of the American control conference, (pp.4045-4047).

2. Åström, K.J. and Furuta, K. (1996) Swinging up a pendulum by energy control, presented at 13th IFAC world congress.

3. Ohsumi A and Izumikawa, T.(1995) Nonlinear control of swing-up and stabilization of an inverted pendulum, proceedings of the 34th conference on decision & control, (pp. 3873-3880).

4. Zhong, W. and Röck, H.(1991) Energy and passivity based control of the double inverted pendulum on a cart, presented at IEEE Conference on Control Applications, 2001.

5. Furuta, K. and Yamakita,(1991) M. Swing up control of inverted pendulum, IECON `91, (pp. 2193-2198).

6. Furutam, K. and Yamakita, M.(1992) Swing-up control of inverted pendulum using pseudo-state feedback, proceedings of Instn. Mech Engrs, **206**: 263-269.

7. Wei Q, et al. (1995) Nonlinear Controller for an Inverted Pendulum having Restricted Travel" Automatica **31**(6): 841-850.

8. Slotine, J.J.E. and Li, W (1991) *Applied Non-Linear Control*, Prentice- Hall Inc., New Jersey.

9. Chung, C. C. and J. Hauser (1995): Nonlinear Control of a Swinging Pendulum. *Automatica*, **31**(6): 851–862

10. Krishnan R (2001), *Electric Motor Drives: Modeling, Analysis, and Control.* Upper Saddle River, NJ: Prentice-Hall.

11. Medrano-Cerda G.A (1999), Robust computer control of an inverted pendulum, IEEE Control Syst. Mag., **19**(3): 58–67.

12. Huang S. J and C. L. Huang (2000), Control of an inverted pendulum using grey prediction model, IEEE Trans. Ind. Appl., **36**(2): 452–458.

13. Fu-Kuei Tsai and Jung-Shan Lin (2003), Nonlinear control design of 360- degree inverted pendulum systems, Proceedings of 4th International Conference on Control and Automation (ICCA'03) (pp 1869-1873), Montreal, Canada.

14. Seiji Yasunobu, Takayuki Iwasaki (2004), Swing-up intelligent control of double inverted pendulum based on human knowledge, Proceedings of SICE Annual Conference Sapporo, Hokkaido Institute of Technology, Japan, (pp 634-638).